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Solvable Ising model on Sierpínski carpets: The partition function

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With a special Sierpínski carpet (SC), the Ising model is exactly solved by a combinatorial approach and graph technique. The rigorous partition function and free energy are obtained and the phase transition is investigated. We argue that the existence of a phase transition strongly depends on the order of ramification of the SC. Our method is extended to deal with other lattices.

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I. INTRODUCTION

Statistical models, such as the Ising model and the Potts model, play an important role in the study of phase transitions and critical phenomena. A key step in solving phase transitions is to calculate the partition function exactly. As we know, finding the rigorous partition function of a statistical model has been a very difficult task; only a few samples, such as the one-dimensional Ising model in the existence of external field and the two-dimensional Ising model without external field (Onsager solution) have been solved exactly so far; meanwhile all efforts concentrate on transitionally invariant lattices.

In the 1980s an object with dilation symmetry, the fractal, has attracted much attention; a good deal of research associated with fractals has been done. In this paper we will only focus on phase transitions and critical phenomena of the Ising model.

Gefen and co-workers [1–3] have written some pioneering works in which they investigated the phase transitions of the Ising and Potts models on Koch curves, Sierpínski gaskets, and Sierpínski carpets (SC's) by means of decimation and Migdal-Kadanoff renormalization-group techniques. They pointed out that the Ising model can be solved exactly on any fractal whose order of ramification is finite, using exact renormalization-group methods. They also led to the following conclusions: there is no phase transition if the order of ramification R equals a finite number and there is a finite-temperature transition if R is infinity. Since then, a lot of work related to the same object has been done [4]; however, an explicit calculation of the partition function and the free energy for the Ising model is still lacking.

In this paper we construct a SC which can simulate some real random fractals, and use the graph method and the combinatorial approach to calculate explicitly the partition function and the free energy. In the following we will describe the method in detail and give some useful discussions and extensions.

II. PARTITION FUNCTION: METHOD OF CALCULATION

First of all we construct a special kind of SC. Let us start with a two-dimensional square and a three-dimensional cube and divide them into 9 equal subsquares and 27 subcubes, respectively; then remove 4 subsquares and 18 subcubes from them as shown in Figs. 1 and 2; repeating the process, we finally acquire the SC embedded in two- and three-dimensional Euclidean spaces. The first stage ($n=1$) of construction is called the generator of SC's. Following the definition, the fractal dimensions of SC's are $d_f=-\ln 5/\ln 2=1.465$ and $d_f=\ln 9/\ln 3=2$, respectively. The order of ramification of SC's given here has a finite number other than infinity; because of this it is possible to calculate exactly the partition function.

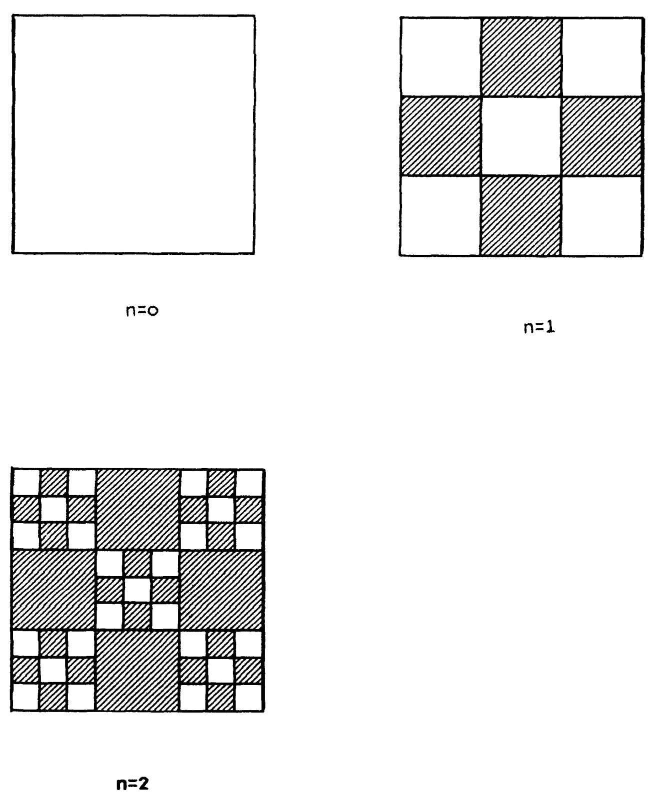

FIG. 1. The process of construction of a Sierpínski carpet embedded in two-dimensional Euclidean space. The shadow parts are removed. The $n=1$ stage of construction is the generator. The fractal dimension $d_f=1.465$.

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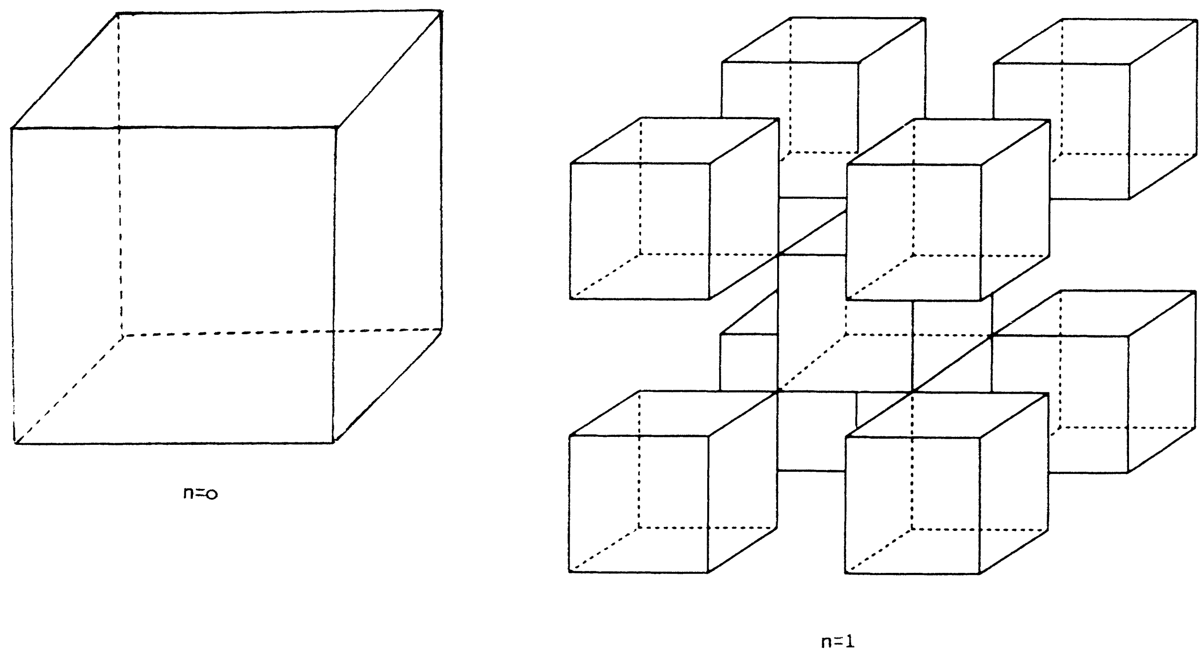


FIG. 2. Similar to Fig. 1. The fractal dimension $d_f = 2$.

We now put the Ising spin on each site of the SC and suppose the existence of an interaction between nearest-neighbor (NN) spins. The reduced model Hamiltonian can be written as

$$-\beta H = K \sum_{\mathrm{NN}} \sigma_i \sigma_j \ , \tag{1}$$

where the Ising spin $\sigma_i = \pm 1$, $K = \beta J = J/kT$ is the reduced coupling parameter, and the summation is over all nearest-neighbor spin pairs. To study phase transitions we calculate the partition function

$$Z = \sum_{\{\sigma\}} e^{-\beta H} \ , \tag{2}$$

where $\{\sigma\}$ denotes all possible spin configurations of the system. In the nth stage of construction of the SC, the partition function can be written as

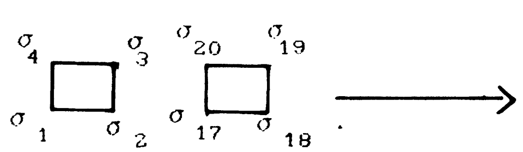
$$\begin{aligned} Z_n &= \sum_{\{\sigma\}} \exp\Big(K \sum_{\mathrm{NN}} \sigma_i \sigma_j\Big) = \sum_{\{\sigma\}} \prod_{\mathrm{NN}} e^{K\sigma_i\sigma_j} = \sum_{\{\sigma\}} \prod_{\mathrm{NN}} (\cosh K)(1+\sigma_i\sigma_j \tanh K) \\ &= (\cosh K)^M \sum_{\{\sigma\}} \prod_{\mathrm{NN}} (1+\sigma_i\sigma_j \tanh K) \ , \end{aligned} \tag{3}$$

where M is the number of edges of the SC.

For convenience of calculation, we first consider the structure of Fig. 1. We expand the product of expression (3):

$$\begin{aligned} \sum_{\{\sigma\}} \prod_{\mathrm{NN}} (1+\sigma_i\sigma_j \tanh K) = \sum_{\{\sigma\}} \Bigg[1 &+ \sum_{\mathrm{NN}} \sigma_i\sigma_j \tanh K + \sum_{\mathrm{NN}} (\sigma_{i_1}\sigma_{j_1})(\sigma_{i_2}\sigma_{j_2}) \tanh^2 K \\ &+ \cdots + \sum_{\mathrm{NN}} (\sigma_{i_1}\sigma_{j_1}) \cdots (\sigma_{i_M}\sigma_{j_M}) (\tanh K)^M \Bigg] \ . \end{aligned} \tag{4}$$

It is not difficult to see that the nonzero contributions to (4) only come from those terms in which the related lattice sites form a number of closed nonbranching graphs; for example, the following graph corresponds to the term



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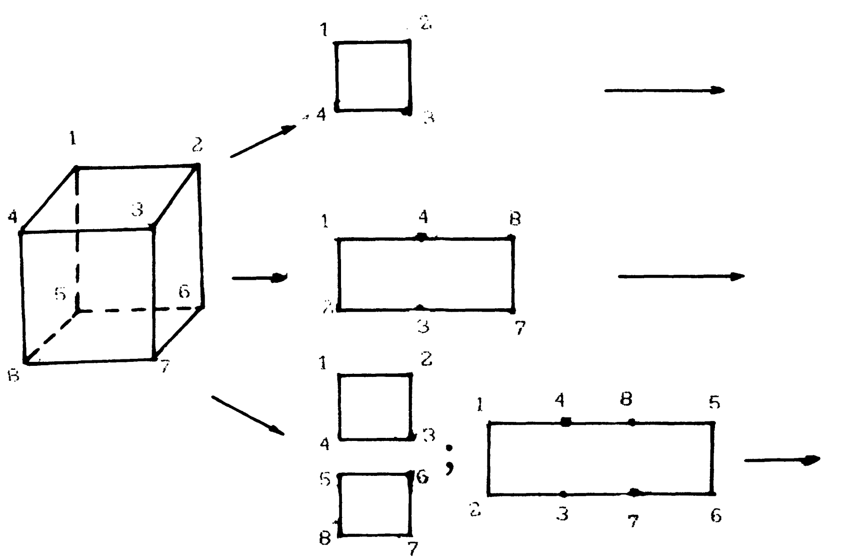
$$\sum_{\{\sigma\}} [(\sigma_1\sigma_2)(\sigma_2\sigma_3)(\sigma_3\sigma_4)(\sigma_4\sigma_1)] \times[(\sigma_{17}\sigma_{18})(\sigma_{18}\sigma_{19})(\sigma_{19}\sigma_{20})(\sigma_{20}\sigma_{17})] \times \tanh^8 K = 2^{N_n} \tanh^8 K \ , \qquad (5)$$

where N_n is the total number of the sites of the fractal lattice. We can find that the basic unit of the closed nonbranching graphs is a subsquare, in which each site has an even number of in and out bonds. In graph theory they are called *even graphs* (Mayer diagrams) [5]. Therefore the result of expression (4) will be composed of all possible combinations of m subsquares; in consequence, we can write the partition function as follows:

$$Z_n = (\cosh K)^{M} 2^{N_n} \left\{ 1 + m \tanh^4 K + \frac{m!}{2!(m-2)!} \tanh^8 K + \cdots + \frac{m!}{1!(m-1)!} \tanh^{41} K + \cdots \right\} = (\cosh K)^{M} 2^{N_n} (1+\tanh^4 K)^m \ , \qquad (6)$$

where m is the total number of subsquares in the nth stage of construction of the SC and $M = 4m$.

Next we investigate the SC embedded in three-dimensional space (Fig. 2). The nonzero contributions to (4) become much more complicated in the underlying system. With the same argument as above, we can display the lower-order nonzero contribution graphs embedded in the subcube as follows:



$\tanh^4 K$,

$\tanh^6 K$, (7)

$\tanh^8 K$,

where a factor $(\tanh K)$ is assigned on each bond. It is clear that the total number of even graphs with factors $(\tanh K)^4$, $(\tanh K)^6$, and $(\tanh K)^8$ in a subcube will be 6, 8, and 11, respectively. As a result, we obtain the rigorous partition function

$$Z_n = (\cosh K)^{M} 2^{N_n} \left\{ 1 + mE(k) + \frac{m!}{2!(m-2)!} E^2(K) + \cdots + \frac{m!}{1!(m-1)!} E^1(K) + \cdots \right\} . \qquad (8)$$

In the present case, $M = 12m$ and the function $E(k)$ is

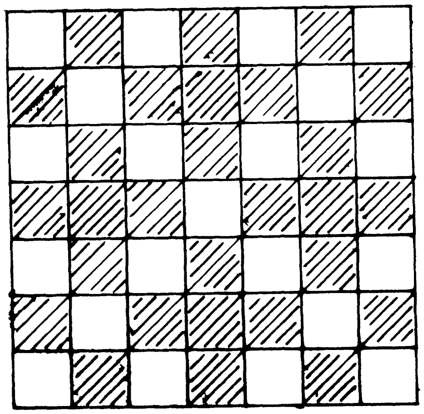
$$E(K) = 6\tanh^4 K + 8\tanh^6 K + 11\tanh^8 K \ , \qquad (9)$$

which represents the nonzero contributions of all possible even graphs in a subcube.

III. FREE ENERGY

We now turn on the exact expression of the free energy per site in the thermodynamic limit; it will be for the structure of Fig. 1:

$$f = \lim_{n\to\infty} \frac{-kT}{N_n} \ln Z_n = \lim_{n\to\infty} \frac{-kT}{N_n} [M \ln(\cosh K) + N_n \ln 2 + m \ln(1+\tanh^4 K)] \ , \qquad (10)$$



n=1

FIG. 3. A generator of a Sierpínski carpet ($n=1$).

where $N_n=3\times 5^n+1$, $m=5^n$, and $M=4\times 5^n$. Finally we get

$$f=-kT[\tfrac{4}{3}\ln(\cosh K)+\ln 2+\tfrac{1}{3}\ln(1+\tanh^4 K)] . \quad (11)$$

Since $\tanh K<1$ and $\cosh K$ is a finite value when $T\neq 0$, the free energy will be an analytical function of $K=J/kT$, which implies that there is no finite-temperature transition, i.e., critical temperature $T_c=0$. Similarly for Fig. 2, we have the free energy per site as follows:

$$\begin{aligned} f&=\lim_{n\to\infty}\frac{-kT}{N_n}\ln Z_n\\ &=\lim_{n\to\infty}\frac{-kT}{N_n}[M\ln(\cosh K)+N_n\ln 2+m\ln E(K)]\\ &=-kT[\tfrac{12}{7}\ln(\cosh K)+\ln 2+\tfrac{1}{7}\ln E(K)] \quad (12)\end{aligned}$$

which is also an analytical function of K and thus no phase transition exists.

As we expect, the zero-temperature transition occurs due to the fact that the Sierpínski carpets mentioned here have a finite order of ramification R [3]. The result can be understood with the standard inequalities or entropy arguments (see, e.g., [6] and [7]). At any finite temperature, the system may break into domains, gaining free energy, and as a result order is destroyed [3].

IV. EXTENSION AND DISCUSSION

The combinatorial approach and graph technique can also be applied to those lattices (they may not be fractal lattices) in which the subunits connect with each other only through a common site. Figures 3 and 4 show such lattices. Figure 4 is a Bethe-type lattice but not fractal; its exact partition function can be expressed as follows:

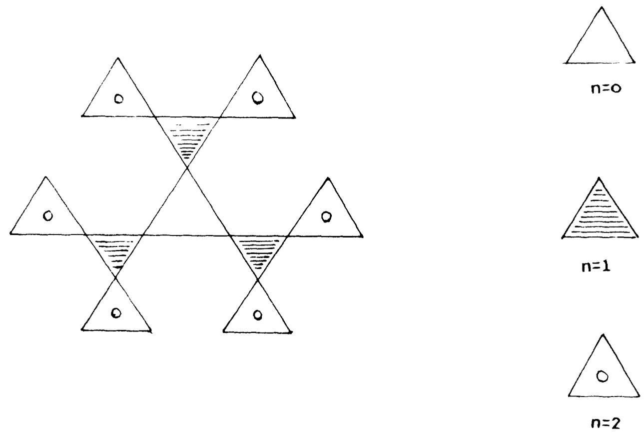


FIG. 4. Bethe-type lattice. n denotes the different generations.

$$Z_n=(\cosh K)^M 2^{N_n}(1+\tanh^3 K)^m . \quad (13)$$

The free energy per site in the thermodynamic limit is

$$\begin{aligned} f&=\lim_{n\to\infty}\frac{-kT}{N_n}\ln Z_n\\ &=-kT[\tfrac{3}{2}\ln(\cosh K)+\ln 2+\tfrac{1}{2}\ln(1+\tanh^3 K)] . \quad (14)\end{aligned}$$

In expression (13) a notable feature appears, i.e., the factor $(\tanh K)^3$ replaces $(\tanh K)^4$ in formula (6), which reflects the different geometric property of a lattice structure. In addition, the second-order derivative of the free energy is continuous for all temperatures, which shows that the Ising model on the Bethe-type lattice does not occur at a finite-temperature phase transition. However, we have noted that the transition will be displayed in the field dependence of the free energy and will become arbitrarily weak [8].

In our model we have never considered the existence of an external field. In fact, if the applied field enters the model Hamiltonian, we will not be able to obtain the rigorous partition function.

ACKNOWLEDGMENTS

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[1] Y. Gefen, A. Aharony, and B. B. Mandelbrot, J. Phys. A **16**, 1267 (1983).
[2] Y. Gefen, A. Aharony, Y. Shapir, and B. B. Mandelbrot, J. Phys. A **17**, 435 (1984).
[3] Y. Gefen, A. Aharony, and B. B. Mandelbrot, J. Phys. A **17**, 1277 (1984).
[4] See, for example, Y. K. Wu and B. Hu, Phys. Rev. A **35**, 1404 (1987); B. Lin and Z. R. Yang, J. Phys. A **19**, L49 (1986); L. Hao and Z. R. Yang, *ibid.* **20**, 1627 (1987).
[5] F. Y. Wu and Z. R. Yang, Prog. Phys. **1**, 314 (1981) (in Chinese).
[6] R. B. Griffiths, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 1.
[7] C. J. Thompson, in *Phase Transitions and Critical Phenomena* (Ref. [6]).
[8] E. Muller-Hartmann and J. Zittartz, Phys. Rev. Lett. **33**, 893 (1974).